# Mathematical Foundations of Infinite-Dimensional Statistical Models

## 2.7 Asymptotics for Extremes of Stationary Gaussian Processes

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2018.09.14

# Asymptotics for Extremes of Stationary Gaussian Processes

- Focus on the limiting distribution of $\max_{1\leq k\leq n}|X_n|$ for Gaussian sequences and of $\sup_{0\leq t\leq T}|X(t)|$ for Gaussian processes on $\mathbb{R}^+$.
- Only theoretical interest because the speed of convergence for limits theorems is very slow (for sequences, of the order of $1/\log n$).
- Here we consider stationary processes.

# Gaussian Sequences

- Theorem 2.7.1 Let $g_i$, $i \leq n$, be independent standard normal random variables. For each $n \geq e^2$, set

$$a_n = (2 \log n)^{1/2}, \quad b_n = a_n - \frac{\log \log n + \log \pi}{2a_n}$$

Then, for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr \left\{ a_n \left( \max_{1 \leq i \leq n} |g_i| - b_n \right) \leq x \right\} = \exp(-e^{-x}). \qquad (2.90)$$

## Gaussian Sequences

- Proof of Theorem 2.7.1 For $g$ standard normal, let $\Phi(x) = \Pr\{|g| \leq x\}$, and let $\phi(x) = (2/\pi)^{1/2} e^{-x^2/2}, x \geq 0$. Then, for $u > 0$,

$$\frac{u}{1+u^2}\phi(u) \leq 1 - \Phi(u) \leq \frac{1}{u}\phi(u).$$

For $\mathbf{x} \in \mathbb{R}$ fixed, set $u_n = x/a_n + b_n$. By definition of $u_n$, we have

$$\frac{n\phi(u_n)}{u_n} \to e^{-x}, \quad \text{as } n \to \infty.$$

Finally,

$$\Pr\left\{\max_{i \leq n} |g_i| \leq u_n\right\} = \Phi^n(u_n) = [1 - (1 - \Phi(u_n))]^n$$
$$\approx e^{-n(1-\Phi(u_n))} \approx e^{-n\phi(u_n)/u_n} \to \exp(-e^{-x}),$$

as $n \to \infty$.

# Gaussian Sequences

- Definition 2.7.2 A sequence $\{\xi_n\}$ of r.v.s is stationary if for any $n_1, \ldots, n_m \in \mathbb{N}$, $m > 0$ and for any $k \geq 0$, the joint probability law of the r.v.s $\xi_{n_1+k}, \ldots, \xi_{n_m+k}$ does not depend on k.
- If $\{\xi_n\}$ is a centred (jointly) Gaussian sequence, then it is stationary iff the covariances $\mathbb{E}(\xi_m \xi_{n+m})$ do not depend on $m$. In this case, $r(n) = \mathbb{E}(\xi_m \xi_{n+m})$ is the covariance function of the sequence.

- Lemma 2.7.3 Let $\{r_j\}$ be a sequence of numbers s.t. $\sup_n |r_n| < 1$ and $r_n \log n \to 0$ as $n \to \infty$, and let $\{u_n\}$ be a sequence of positive constants s.t. $\sup_n n(1 - \Phi(u_n)) < \infty$. Then

$$\lim_{n\to\infty} n \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_n^2}{1+|r_j|}\right) = 0. \tag{2.93}$$

- Theorem 2.7.4 Let $\{\xi_i\}_{n=1}^{\infty}$ be a stationary sequence of standard normal random variables s.t. its sequence of covariances $r_n = \mathbb{E}(\xi_m \xi_{n+m})$ satisfies $r_n \log n \to 0$. For each $n \geq e^2$, set

$$a_n = (2\log n)^{1/2}, \quad b_n = a_n - \frac{\log\log n + \log \pi}{2a_n}$$

Then, for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left\{a_n(\max_{1\leq i\leq n} |\xi_i| - b_n) \leq x\right\} = \exp(-e^{-x}). \tag{2.96}$$

*Proof* By Thm. 2.4.7, Lemma 2.7.3,

$$\left|\Pr\left\{a_n(\max_{1\leq i\leq n} |\xi_i| - b_n) \leq x\right\} - \Pr\left\{a_n(\max_{1\leq i\leq n} |g_i| - b_n) \leq x\right\}\right| \leq \frac{2}{\pi} n \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_n^2}{1+|$$

## Gaussian Processes Indexed by $[0, \infty)$

- A stocahstic process $\xi(t)$, $t \geq 0$, is *stationary* if the finite-dimensional marginal distributions of $\xi(t_1 + s), \ldots, \xi(t_n + s)$ do not depend on $s$, for all $n$ and for any $s$ s.t. $t_i + s \geq 0, i = 1, \ldots, n$.
- If $\xi(t)$, $t \geq 0$ is a centred GP, then it is stationary iff the covariances $\mathbb{E}(\xi(s)\xi(s + t))$ do not depend on $s$. In this case, $r(t) = \mathbb{E}(\xi(0)\xi(t))$ is the covariance function of the GP.
- A stationary GP is *normalized* if $\xi(t)$, $t \geq 0$ is standard normal.

# Gaussian Processes Indexed by $[0, \infty)$

- Now let $\xi(t)$, $t \in [0, \infty)$, be a normalized stationary GP with continuous sample paths and with covariance $r(t) = \mathbb{E}(\xi(s)\xi(s+t))$ satisfying, for some $\alpha \in (0, 2]$ and $C \in (0, \infty)$,

$$r(t) < 1, \text{ for } t > 0 \text{ and } r(t) = 1 - C|t|^{\alpha} + o(|t|)^{\alpha}, \text{ as } t \to 0. \quad (2.97)$$

At the end of this section, for some $a_T$ and $b_T$, we want to get

$$\lim_{T \to \infty} \Pr\left\{ a_T( \sup_{t \in [0, T]} |\xi(t)| - b_T) \leq x \right\} = \exp(-e^{-x}),$$

for all $x \in \mathbb{R}$.

# Gaussian Processes Indexed by $[0, \infty)$

- The proof is divided into three parts:
  1. A limit theorem for the high excursions of the process over a fixed finite interval;
  2. The process is replaced by its absolute value;
  3. The limit theorem for an interval increasing to infinity.

## Gaussian Processes Indexed by $[0, \infty)$

- Let $\bar{\Phi}(u) = (2\pi)^{-1/2} \int_u^\infty e^{x^2/2} dx$ be the tail probability function for the standard normal distribution.
- For each $\alpha \in (0, 2]$, $\zeta(t)$ is a (auxiliary) nonstationary GP with mean $-|t|^\alpha$ and covariance $r_\zeta(s, t) = t^\alpha + s^\alpha - |t - s|^\alpha$.
- Theorem 2.7.5 Let $\xi(t)$, $t \in [0, \infty)$, be a normalized centred stationary GP with continuous sample paths and with covariance $r(t)$ satisfying condition (2.97) for some $\alpha \in [0, 2]$ and $C \in (0, \infty)$. Then, for any $h \in (0, \infty)$,

$$\lim_{u \to \infty} \frac{1}{u^{2/\alpha} h \bar{\Phi}(u)} \Pr\left\{ \sup_{t \in [0,h]} \xi(t) > u \right\} = C^{1/\alpha} H_\alpha, \tag{2.99}$$

where $H_\alpha = \lim_{a \downarrow 0} \frac{1}{a} \Pr\left\{ \sup_{k \in \mathbb{N}} \zeta(ak) + \eta \leq 0 \right\} \in (0, \infty)$.

- Corollary 2.7.6 Under the assumptions of Thm. 2.7.5,

$$\lim_{u \to \infty} \frac{1}{u^{2/\alpha} h \bar{\Phi}(u)} \Pr \left\{ \sup_{t \in [0,h]} |\xi(t)| > u \right\} = 2C^{1/\alpha} H_\alpha. \qquad (2.108)$$

## Gaussian Processes Indexed by $[0, \infty)$

- Now we obtain a limit theorem for the interval $[0, h]$ increasing to infinity.
- Proposition 2.7.8 Let $\xi(t)$, $t \in [0, \infty)$, be a normalized centred stationary GP with continuous sample paths and with covariance $r(t)$ satisfying $r(t) < 1$, for $t > 0$, $r(t) = 1 - C|t|^{\alpha} + o(|t|^{\alpha})$, for some $\alpha \in (0, 2]$ as $t \to 0$, and $r(t) \log t \to 0$, as $t \to \infty$. Let $\tau > 0$, and for each $T$, let $u = u_T$ be s.t. condition $T\mu(u_T) \to \tau$, as $T \to \infty$. Then

$$\lim_{T \to \infty} \Pr\left\{ \sup_{0 \le t \le T} |\xi(t)| \le u_T \right\} \to e^{-\tau}.$$

# Gaussian Processes Indexed by $[0, \infty)$

- Theorem 2.7.9 Let $\xi(t)$, $t \in [0, \infty)$, be a normalized centred stationary GP with continuous sample paths and with covariance $r(t)$ satisfying $r(t) < 1$, for $t > 0$, $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$, for some $\alpha \in (0, 2]$ as $t \to 0$, and $r(t) \log t \to 0$, as $t \to \infty$. Set $a_T = (2 \log T)^{1/2}$ and

$$b_T = a_T + \frac{(2-\alpha)/\alpha \log \log T + 2 \log((2\pi)^{-1/2} 2^{(2+\alpha)/\alpha} C^{1/\alpha} H_\alpha)}{2a_T}. \tag{2.120}$$

Then, for all $x \in \mathbb{R}$,

$$\lim_{T \to \infty} \Pr\left\{ a_T( \sup_{t \in [0,T]} |\xi(t)| - b_T) \leq x \right\} = \exp(-e^{-x}).$$

# Notes

- A centred GP $Y(t)$, $t \in \mathbb{R}$, is cyclostationary if its covariance function $t \mapsto r(t, t+v) := \mathbb{E}(Y(t)Y(t+v))$ is periodic in $t$ for every $v \in \mathbb{R}$ with period independent of $v$.

- Theorem 2.8.3 Let $X(t)$, $t \in \mathbb{R}$ be a cyclostationary, centred GP with period $1$, variance $\sigma_X(t)$ and covariance $r_X(s, t)$. Under the some conditions, for all $x \in \mathbb{R}$, we have

$$\lim_{T\to\infty} \Pr\left\{ a_T \left( \sup_{t\in[0,T]} |X(t)| - b_T \right) \leq x \right\} = \exp(-e^{-x}),$$

where $a_T = \sqrt{2 \log T}$ and

$$b_T = a_T - \frac{\log\log T + \log \pi - \log(1 - \frac{\mathbb{E}(X'(t_0))^2}{\sigma_X''(t_0)})}{2a_T}.$$